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Renormalization in non-relativistic quantum mechanics

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Abstract. The importance and usefulness of renormalization are emphasized in non-relativistic quantum mechanics. The momentum space treatment of both two-body bound state and scattering problems involving some potentials singular at the origin exhibits ultraviolet divergence. The use of renormalization techniques in these problems leads to finite converged results for both the exact and perturbative solutions. The renormalization procedure is carried out for the quantum two-body problem in different partial waves for a minimal potential possessing only the threshold behaviour and no form factors. The renormalized perturbative and exact solutions for this problem are found to be consistent with each other. The useful role of the renormalization group equations for this problem is also pointed out.

1. Introduction

The ultraviolet divergences in perturbative quantum field theory can be eliminated in many cases by renormalization to define physical observables, such as charge or mass, which are often termed the physical scale(s) of the problem [1–3]. Ultraviolet divergences appear in exact as well as perturbative treatments of the non-relativistic quantum mechanical two-body problem in momentum space interacting via two-body potentials with certain singular behaviour at short distances [4–10] in two and three space dimensions. Renormalization of the potential model leads to a scale(s) and a finite physical observable(s) [6].

Renormalization of a physical quantum mechanical model is essential in reproducing experimental results irrespective of whether the original model exhibits ultraviolet divergence or not. Renormalization removes the effect of different uncertainties and approximations of a physical model on the observables and brings some of the theoretical predictions in agreement with experiment. Such uncertainties exist in all quantum mechanical models. Even in the most well-understood quantum mechanical hydrogen-atom problem only the long distance behaviour of the Hamiltonian can be considered to be known. For distances smaller than the radius of the proton, the electron–proton potential is not the bare Coulomb potential but some regularized Coulomb potential which, unlike the original Coulomb potential, does not diverge and leads to a constant value as the electron–proton separation r goes to zero. Also at this scale the effect of field theoretic corrections to the Hamiltonian is relevant. The detailed behaviour of this regularized Hamiltonian for small r depends on the charge distribution of proton and is not usually known. The role of renormalization is to remove the uncertainty of the regularized potential by fixing some of the observable(s). The effect of renormalization in the hydrogen atom problem is small and

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not evident as the radius of the proton is very small. The effect is indispensable in large atoms and specially in muonic atoms where the orbit of the muon could have a significant overlap with nuclear matter.

In the above-mentioned Coulomb problem the potential is divergent at $r = 0$. In spite of the singularity of the Coulomb potential at $r = 0$, both the scattering and bound state problems with the original potential are solvable and do not produce ultraviolet divergences. The role of renormalization in this problem is to introduce a regularized well-defined potential which reproduces some of the observables.

The situation is different for potentials with a stronger divergence at $r = 0$ than the Coulomb potential. These are the potentials which lead to the above-mentioned ultraviolet divergences in momentum space. If these divergent potentials are attractive at $r = 0$, the original problem does not permit convergent solution in either momentum or configuration space. The bound state problem collapses and produces an infinite number of bound states with an accumulation point at infinite binding. The scattering Lippmann–Schwinger equation for these potentials possesses a non-compact kernel and hence is not amenable to numerical solution. A finite and meaningful physical solution is obtained only after renormalization. If the divergent potentials are repulsive at $r = 0$, in the configuration space treatment one can obtain a finite convergent solution essentially by imposing some constraints, such as the solution should vanish at some small r . In this way the trouble with integration over the singular potential near $r = 0$ is avoided. In the case of many repulsive divergent potentials, this procedure works and produces physically meaningful results, some examples being the repulsive soft and hard core potentials exhibiting ultraviolet divergences. Even in these cases the momentum space treatment, after an appropriate truncation of the Hamiltonian at small r , may lead to a Lippmann–Schwinger equation with non-compact kernel. Renormalization is then necessary to produce finite and physically meaningful results.

The usual difficulty in momentum space treatment with the Coulomb potential is the large distance or the infrared divergence. This could be avoided with the usual Yukawa potential in nuclear and atomic physics. The Yukawa potential possesses the same large momentum or short distance behaviour as the Coulomb potential but no infrared divergence. For a Yukawa potential without ultraviolet (and infrared) divergence(s), renormalization improves the large momentum or short distance convergence properties. In this problem renormalization is not necessary but is only desirable. Renormalization makes this potential smoother and hence easier for numerical and analytic treatment. However, renormalization is indispensable for problems with ultraviolet divergence in momentum space.

In this work we shall be limited to the study of renormalization of the three-dimensional two-body problem possessing ultraviolet divergence in momentum space in close analogy with field theoretic problems. An account of parts of this work has recently appeared [5,6]. Most of the present ideas can also be used in two dimensions and in configuration space treatments [5,8]. We illustrate the present procedure for a minimal potential in different partial waves. In momentum space this potential possesses only the threshold behaviour and is given by $V_L(p', p) = p'^L \lambda p^L$, where L is the angular momentum. As the scattering Lippmann–Schwinger equation has the same generic form for all partial waves, the ultraviolet divergence of this potential model becomes stronger and stronger as L increases. The leading ultraviolet divergence of the momentum space integrals, encountered while solving the Lippmann–Schwinger equation with this potential, is linear (cubic, . . .) in nature for $L = 0$ (1, . . .). The renormalization of this potential model can be performed by fixing at least one observable, or equivalently, by introducing at least one physical scale. It is also possible to renormalize by introducing more than one physical scale. We renormalize both

the exact and the perturbative solutions and find that the renormalized exact and perturbative solutions are consistent with each other.

We also derive the renormalization group (RG) equations for this problem. These equations clearly exhibit the important scaling behaviours of the different renormalized solutions. The RG equations can be written for the scattering solutions expressed in terms of certain physical scales closely related to scattering observables. These equations are valid in general, independent of the existence of ultraviolet divergence in the original problem. Such RG equations and the associated scaling behaviour involving observables are interesting from a physical point of view.

The ultraviolet divergence of the present problem for $L = 0$ can be compared to the ultraviolet structure and high-energy behaviour of the $\lambda\phi^4$ field theory [2, 8, 10]. The super-renormalizable $\lambda\phi^4$ field theory in $1 + 1$ dimensions possesses ultraviolet logarithmic divergence, requires regularization, and is perturbatively renormalizable [2, 4]. The non-relativistic scattering problem with contact interaction in two dimensions also has similar logarithmic divergence [9]. The renormalizable $\lambda\phi^4$ field theory in $3 + 1$ dimensions has both logarithmic and quadratic divergences [2, 4]. We have verified that the non-relativistic scattering problem with the present minimal potential in two dimensions also has similar logarithmic and quadratic divergences. In the present study of scattering in three dimensions, although the divergent terms are of different nature, the renormalization can be performed in a similar fashion. In the field theoretic problem, one cannot go beyond few lowest orders of perturbation theory. On the other hand, the non-relativistic scattering problem with the present minimal potential possesses stronger ultraviolet divergences than in the $\lambda\phi^4$ field theory and can be solved to find both the exact and the perturbative solutions analytically. In the present work we find that the exact renormalized solution is consistent with the perturbative one. The study of the present analytic model will allow us to understand most of the subtleties of renormalization and RG equations.

The plan of our work is as follows. In section 2 we perform the renormalization of the exact solution for the minimal potential in different partial waves. In section 3 the renormalization of the perturbative solution is carried out and consistency of the renormalized exact and perturbative solutions is demonstrated. In section 4 we derive the RG equations and discuss the scaling properties of the renormalized solution. In section 5 a brief summary of the present work is presented.

2. Renormalization of the exact solution

The partial-wave Lippmann–Schwinger equation for the scattering amplitude $T_L(p, q, k^2)$ in three dimensions, at centre of mass energy k^2 , is given by

$$T_L(p', p, k^2) = V_L(p', p) + \frac{2}{\pi} \int q^2 dq V_L(p', q) G(q; k^2) T_L(q, p, k^2) \quad (1)$$

with the free Green function $G(q; k^2) = (k^2 - q^2 + i0)^{-1}$, in units $\hbar = 2m = 1$, where m is the reduced mass.

We discuss potential scattering with the minimal potential in different partial waves. The present minimal potential in the L th partial wave is taken to be $V_L(p', p) = p'^L \lambda p^L$, which is the usual δ potential for $L = 0$. For increasing L this potential presents stronger and stronger ultraviolet divergence. The reason for studying this potential is that it is analytically tractable and presents arbitrarily strong ultraviolet divergence as L increases. It is not *a priori* clear that potentials with arbitrarily strong ultraviolet divergence can be meaningfully renormalized. Physically, this potential is one of arbitrary short range in

higher partial waves and should be compared with the S wave δ function potential. If a meaningful solution of the problem could be found they could be of use in different areas of physics where the details of a potential is not of concern. (a) The renormalized solution of the minimal potential could be used in problems of statistical mechanics, such as, in Cooper pairing in superconductivity. There is evidence of pairing in higher partial waves [12]. In this case the details of the phonon induced short-range electron–electron potential is irrelevant. The Cooper and the Bardeen–Cooper–Schrieffer (BCS) equations in superconductivity, have been satisfactorily renormalized in S wave, but not in higher partial waves [12]. The present work should be of relevance to the renormalization of the Cooper and BCS equations in higher partial waves. (b) The present renormalization scheme is also of interest in deriving a nucleon–nucleon potential from an effective field theory [13] as suggested by Weinberg [5]. In this derivation one needs to sum an infinite series of Feynmann diagrams. In the lowest order, the nucleon–nucleon potential as derived from the effective field theory includes an attractive delta potential, the solution of which has been successfully renormalized [13]. However, in higher order one obtains a potential with stronger divergence involving powers of momentum (see, for example (3.4) of [14]). Such a potential should be renormalizable following the scheme presented here.

The present approach is also applicable to other potentials with weaker ultraviolet divergences and/or permitting only numerical solution. A numerical study of the renormalization scheme has recently been made [15].

For the above mentioned minimal potential, the t matrix of (1) permits the following analytic solution

$$T_L(p', p, k^2) = p'^L \tau_L(k) p^L \quad (2)$$

with the τ function defined by

$$\tau_L(k) = [\lambda^{-1} - I_L(k)]^{-1} \quad (3)$$

$$I_L(k) = (2/\pi) \int q^2 dq q^{2L} G(q; k^2). \quad (4)$$

As the τ function completely determines the t matrix, we shall consider only the renormalization of the τ function. Here the condition of unitarity is given by

$$\Im T_L(k) = -k |T_L(k)|^2 \quad (5)$$

where $T_L(k) = T_L(k, k, k^2)$ and \Im denotes the imaginary part.

The integral $I_L(k)$ of (4) possesses ultraviolet divergence. For $L = 0$ (1, . . .) the leading divergence of this integral is linear (cubic, . . .) in nature. A finite result for the t matrix of (2) and (3) can be obtained only if λ^{-1} also diverges in a similar fashion and cancels the divergence of $I_L(k)$. The function $\lambda I_L(k)$ is the trace of the kernel of the integral equation (1) and possesses ultraviolet divergence. The kernel of (1) is non-compact and it does not have a scattering solution.

Hence some regularization is needed to give meaning to (1). This can be achieved by using a regularized Green function involving a cut-off. One example is the following regularized Green function with a smooth cut-off Λ for $L = 0$ as in [6]

$$G_R(q, \Lambda; k^2) = (k^2 - q^2 + i0)^{-1} + (\Lambda^2 + q^2)^{-1} = \frac{k^2 + \Lambda^2}{(k^2 - q^2 + i0)(\Lambda^2 + q^2)}. \quad (6)$$

However, in the present work we shall use the following regularized Green function with a sharp cut-off

$$G_R(q, \Lambda; k^2) = (k^2 - q^2 + i0)^{-1} \Theta(q - \Lambda) \quad (7)$$

$\Theta(x) = 0$ for $x > 0$ and $= 1$ for $x < 0$. In (6) and (7) $\Lambda (\gg k)$ is a large but finite quantity. The reason for choosing the Green function (7) is that it is equally applicable for all types of ultraviolet divergences in all partial waves, whereas Green function (6) is only valid for a linear divergence as encountered in the $L = 0$ treatment of [6] and requires modification if the divergence is stronger as in this work. Though we use the Green function (7) with the minimal potential in the present treatment, the present idea of renormalization can be extended to other (singular) potentials and to other regularized Green function(s). The imaginary part of the Green function is unaffected by this type of regularization, and this guarantees the unitarity condition (5).

In the end, the limit $\Lambda \rightarrow \infty$ has to be taken, which will reduce the regularized Green function to the free Green function. Finite results for physical magnitudes, as $\Lambda \rightarrow \infty$, are obtained only if the coupling λ is also replaced by the so-called bare coupling $\lambda_L(k, \Lambda)$. The choice of the bare coupling is different for different L and can be found by inspection of the following regularized expressions of the integral $I_L(k)$ of (4) for different L

$$I_{RL}(k, \Lambda) \equiv \frac{2}{\pi} \int q^2 dq q^{2L} G_R(q, \Lambda; k^2) \quad (8)$$

$$= -\frac{2}{\pi} \left[\Lambda + \frac{k}{2} \ln \left| \frac{\Lambda - k}{\Lambda + k} \right| \right] - ik \quad L = 0 \quad (9)$$

$$= -\frac{2}{\pi} \left[\frac{\Lambda^3}{3} + k^2 \Lambda + \frac{k^3}{2} \ln \left| \frac{\Lambda - k}{\Lambda + k} \right| \right] - ik^3 \quad L = 1 \quad (10)$$

$$= -\frac{2}{\pi} \left[\frac{\Lambda^5}{5} + \frac{k^2 \Lambda^3}{3} + k^4 \Lambda + \frac{k^5}{2} \ln \left| \frac{\Lambda - k}{\Lambda + k} \right| \right] - ik^5 \quad L = 2. \quad (11)$$

Consistent with the large $\Lambda (\gg k)$ limit, the logarithmic terms in the above expressions for $I_{RL}(k, \Lambda)$ tend to zero, and for a general L we have in this limit

$$I_{RL}(k, \Lambda) = -\frac{2}{\pi} \sum_{i=0}^L \frac{k^{2(L-i)} \Lambda^{2i+1}}{2i+1} - ik^{2L+1}. \quad (12)$$

All the terms in the summation in (12) diverges as $\Lambda \rightarrow \infty$. Except for $L = 0$, these divergent terms are momentum (k) dependent. In the present work, the leading divergence is much stronger for a general L compared to the S wave case. In (12) the leading divergence is like Λ^{2L+1} . The S wave treatment of the δ potential in [6] had only an energy independent term diverging linearly as Λ . For a finite $k (\ll \Lambda)$, the stronger divergence and the energy dependence of the divergent terms in the present case do not introduce any complication and the ideas of [6] can be generalized.

In order to obtain a finite renormalized τ function, the coupling λ should be replaced by the so-called bare coupling defined, for example, by

$$\lambda_L^{-1}(k, \Lambda) = -\frac{2}{\pi} \sum_{i=0}^L \frac{k^{2(L-i)} \Lambda^{2i+1}}{2i+1} - \Lambda_{0L}(k^2) \quad (13)$$

where the function $\Lambda_{0L}(k^2)$ defines the physical scale(s) of the system and characterizes the interaction. In the end the physical scale(s) in $\Lambda_{0L}(k^2)$ should be identified with a physical observable(s). If the problem is characterized by a single physical scale, for example the scattering length a_L , it is appropriate to take $\Lambda_{0L}(k^2)$ to be independent of k^2 : $\Lambda_{0L}(k^2) = -1/a_L^{2L+1}$. If the problem is characterized by two physical scales, such as a scattering length a_L and another physical scale b_L , it is natural to take the following expansion

$$\Lambda_{0L}(k^2) = -1/a_L^{2L+1} - b_L^{1-2L} k^2. \quad (14)$$

We have taken both the scales a_L and b_L to have the dimension of length. A third scale c_L can be accommodated similarly through

$$\Lambda_{0L}(k^2) = -1/a_L^{2L+1} - b_L^{1-2L}k^2 - c_L^{3-2L}k^4 \quad (15)$$

where c_L has also been chosen to have the dimension of length. Equation (15) is just a Taylor series expansion of $\Lambda_{0L}(k^2)$ at low energies. It is realized that in the present renormalization the number of divergent terms and the number of scales are not related.

The regularized τ function of (3) can now be rewritten as

$$\tau_L(k, \Lambda) = [\lambda_L^{-1}(k, \Lambda) - I_{RL}(k, \Lambda)]^{-1} \quad (16)$$

where for a finite Λ , $I_{RL}(k, \Lambda)$ is a convergent integral. As $\Lambda \rightarrow \infty$, however, this integral develops the original ultraviolet divergence. In this limit, the quantity $\lambda_L^{-1}(\Lambda, k)$ of (13) has the appropriate divergent behaviour, that cancels the divergent parts of $I_{RL}(k, \Lambda)$. The explicit dependence of the τ function on Λ has been introduced in (16).

Next the limit $\Lambda \rightarrow \infty$ has to be taken in (16). With this regularization, the renormalized τ function can be written as

$$\tau_{RL}(k, \lambda_R(k, \mu), \mu) = [\lambda_{RL}^{-1}(k, \mu) - I_{RL}(k, \mu)]^{-1} \quad (17)$$

where μ is the scale of the problem and emerges as a result of renormalization. The renormalization scale μ should be contrasted with the physical scale(s) in $\Lambda_{0L}(k^2)$. The renormalized τ function will be independent of μ . In (17) the explicit dependence of the τ function on both μ and the renormalized coupling $\lambda_{RL}(k, \mu)$ has been exhibited. The limiting procedure implied by $\Lambda \rightarrow \infty$ in (16) leads to the following definition for the renormalized coupling $\lambda_{RL}(k, \mu)$

$$\lambda_{RL}^{-1}(k, \mu) = \lim_{\Lambda \rightarrow \infty} [\lambda_L^{-1}(k, \Lambda) - \{I_{RL}(k, \Lambda) - I_{RL}(k, \mu)\}]. \quad (18)$$

In (18), if the limit $\Lambda \rightarrow \infty$ taken, we get

$$\lambda_{RL}(k, \mu) = \lambda_L(k, \Lambda = \mu). \quad (19)$$

Relation (19) between the renormalized coupling and bare coupling depends on the regularization scheme used. Equations (13) and (19) lead to the following expression for the renormalized coupling

$$\lambda_{RL}^{-1}(k, \mu) = -\frac{2}{\pi} \sum_{i=0}^L \frac{k^{2(L-i)} \mu^{2i+1}}{2i+1} - \Lambda_{0L}(k^2). \quad (20)$$

The renormalized coupling for two renormalization scales μ and μ_0 are related by the following flow equation

$$\lambda_{RL}^{-1}(k, \mu) + \frac{2}{\pi} \sum_{i=0}^L \frac{k^{2(L-i)} \mu^{2i+1}}{2i+1} = \lambda_{RL}^{-1}(k, \mu_0) + \frac{2}{\pi} \sum_{i=0}^L \frac{k^{2(L-i)} \mu_0^{2i+1}}{2i+1}. \quad (21)$$

For a general L , this flow equation is energy dependent but independent of the regularization scheme. For $L = 0$, as in [6], the renormalized coupling and the flow equations are energy independent. The absolute value of the renormalized coupling $\lambda_{RL}(k, \mu)$ increases with μ . Thus if we start with a small $\lambda_{RL}(k, \mu_0)$ at a given renormalization scale μ_0 , the effective coupling constant increases with μ as in the $\lambda\phi^4$ model [2]. With the increase of μ one can reach a large enough $\lambda_{RL}(k, \mu)$, where perturbative treatment is not valid. The energy dependence of the renormalized coupling (20) and the flow equation (21) for $L \neq 0$ does not create any complication and one can renormalize the results and write the RG equations.

The present scattering model permits analytic solutions for all L . The renormalized τ function is given by

$$\tau_{RL}(k, \lambda_{RL}(k, \mu), \mu) = \left[\lambda_{RL}^{-1}(k, \mu) + \frac{2}{\pi} \sum_{i=0}^L \frac{k^{2(L-i)} \mu^{2i+1}}{2i+1} + ik^{2L+1} \right]^{-1}. \quad (22)$$

Explicitly, using the renormalized coupling (20), the renormalized τ function can be written as

$$\tau_{RL}(k, \lambda_{RL}(k, \mu), \mu) = [ik^{2L+1} - \Lambda_{0L}(k^2)]^{-1}. \quad (23)$$

This τ function depends on the renormalized coupling $\lambda_{RL}(k, \mu)$, but not on μ , that is the explicit and implicit (through $\lambda_{RL}(k, \mu)$) dependences of the τ function on μ cancel. The physics is determined by the value of $\lambda_{RL}(k, \mu)$ at an arbitrary value of μ [9], or the following μ independent quantity

$$\lambda_{RL}^{-1}(k, \mu) + \frac{2}{\pi} \sum_{i=0}^L \frac{k^{2(L-i)} \mu^{2i+1}}{2i+1} = -\Lambda_{0L}(k^2) \quad (24)$$

as can be seen from (20) and (22).

From (23) we find that the renormalized τ function is a function of $\Lambda_{0L}(k^2)$. It is convenient to express the renormalized τ function in terms of the physical scales a_L , b_L , and c_L introduced in (15). Once this is done, τ_{RL} is determined by the physical scale(s) which are closely related to the observables of the system. Then (23) reduces to

$$\tau_{RL}(k, a_L, b_L, c_L) \equiv \tau_{RL}(k, \lambda_{RL}(k, \mu), \mu) = [ik^{2L+1} + 1/a_L^{2L+1} + b_L^{1-2L}k^2 + c_L^{3-2L}k^4]^{-1}. \quad (25)$$

The name ‘physical scale’ given to a_L , b_L and c_L , is justified as these quantities are a measure of low-energy scattering in each partial wave.

We have here renormalized a divergent physical problem and obtained the well-defined solution (25). As the original problem is ill defined, it is interesting to ask if this renormalized solution is physically acceptable or is just a finite answer obtained by a mathematical trick from the unregularized original problem. The fact, that the renormalized result is physically motivated, can now be established by a careful examination. For $L = 0$ (25) is just the usual effective range expansion for the t matrix [16]. The same is also true for higher partial waves [16]. Hence the renormalized solution (25) is the physically expected solution of the problem for a short-range potential and should lead to acceptable results for other observables. So the present renormalization scheme for $L \neq 0$ should be considered as a natural generalization of our previous results presented in [6].

To bring further evidence to the acceptability of the present result and to demonstrate the self-consistency of the present renormalization scheme, we perform perturbative renormalization of the same problem in the next section and establish the equivalence between the two approaches.

3. Perturbative renormalization

In the last section we performed the renormalization of the exact analytic solution. In the simplest field theoretic $\lambda\phi^4$ model the exact solution is not known because of the creation and annihilation of particles and also because of the quartic nature of the interaction. In that case, one usually performs perturbative renormalization. Though it is expected that the result of perturbative renormalization should be consistent with that of exact renormalization, there

is no general proof in this regard. As the present problem is much simpler than the $\lambda\phi^4$ model, it is also illustrative to perform perturbative renormalization of the present problem and to show that the result is consistent with the exact renormalization of the last section. This consistency can also be established in the case of the exactly soluble Schwinger model for massless quantum electrodynamics in 1 + 1 dimensions.

From (3) the perturbative solution of the present problem is given by

$$\tau_L(k) = \lambda[1 + \lambda I_{RL}(k, \Lambda) + \lambda^2 I_{RL}^2(k, \Lambda) + \lambda^3 I_{RL}^3(k, \Lambda) + \dots] \quad (26)$$

where we have used the regularized version of the integral $I_{RL}(k, \Lambda)$ given by (12).

Up to the first order in perturbation theory in (26), we take $\lambda_L^{(1)} = \bar{\lambda}_L$ and consequently to this order in the redefined coupling strength $\bar{\lambda}_L$, $\tau_L^{(1)}(k) = \bar{\lambda}_L$ and there is no divergence as $\Lambda \rightarrow \infty$. In order to find the result finite to first order one could have taken $\lambda_L^{(1)} = \lambda$. But if we would like to obtain finite results in all orders of perturbation theory, which are consistent with the exact renormalized result of the last section, a different finite coupling ($\bar{\lambda}_L$) has to be introduced in all orders as in this section.

The second order τ function with this $\bar{\lambda}_L$, given by,

$$\tau_L^{(2)}(k) = \bar{\lambda}_L[1 + \bar{\lambda}_L I_{RL}(k, \Lambda)] \quad (27)$$

however, diverges as $\Lambda \rightarrow \infty$, because in this limit the regularized integral $I_{RL}(k, \Lambda)$ of (12) diverges. A finite result for the τ function up to the second order in $\bar{\lambda}_L$ could be obtained by employing the following coupling

$$\lambda_L^{(2)} = \bar{\lambda}_L \left[1 + \bar{\lambda}_L \frac{2}{\pi} \sum_{i=0}^L \frac{k^{2(L-i)} \Lambda^{2i+1}}{2i+1} \right]. \quad (28)$$

With this modified coupling, the second-order τ function (27) is given by

$$\tau_L^{(2)}(k) = \lambda_L^{(2)}[1 + \lambda_L^{(2)} I_{RL}(k, \Lambda)] \quad (29)$$

where $I_{RL}(k, \Lambda)$ is given by (12). With $\lambda_L^{(2)}$ given by (28), the second-order τ function, $\tau_L^{(2)}(k)$, contains terms up to the fourth order in $\bar{\lambda}_L$. Up to the second order in $\bar{\lambda}_L$, $\tau_L^{(2)}(k)$ is finite in the limit $\Lambda \rightarrow \infty$ and is given by

$$\tau_L^{(2)}(k) = \bar{\lambda}_L[1 - \bar{\lambda}_L i k^{2L+1}]. \quad (30)$$

With the second-order coupling constant given by (28), the third-order τ function

$$\tau_L^{(3)}(k) = \lambda_L^{(2)}[1 + \lambda_L^{(2)} I_{RL}(k, \Lambda) + \lambda_L^{(2)2} I_{RL}^2(k, \Lambda)] \quad (31)$$

diverges in the limit $\Lambda \rightarrow \infty$. In order to obtain a finite τ function in this limit up to the third order in $\bar{\lambda}_L$, one could employ the following third-order λ

$$\lambda_L^{(3)} = \bar{\lambda}_L \left[1 + \bar{\lambda}_L \frac{2}{\pi} \sum_{i=0}^L \frac{k^{2(L-i)} \Lambda^{2i+1}}{2i+1} + \bar{\lambda}_L^2 \left(\frac{2}{\pi} \sum_{i=0}^L \frac{k^{2(L-i)} \Lambda^{2i+1}}{2i+1} \right)^2 \right] \quad (32)$$

in the following expression for the third-order τ function

$$\tau_L^{(3)}(k) = \lambda_L^{(3)}[1 + \lambda_L^{(3)} I_{RL}(k, \Lambda) + \lambda_L^{(3)2} I_{RL}^2(k, \Lambda)]. \quad (33)$$

This third-order τ function now contains terms up to the sixth order in $\bar{\lambda}_L$. If the third-order τ function is truncated up to third-order terms in $\bar{\lambda}_L$, and the limit $\Lambda \rightarrow \infty$ is taken, we obtain

$$\tau_L^{(3)}(k) = \bar{\lambda}_L[1 - \bar{\lambda}_L i k^{2L+1} + \bar{\lambda}_L^2 (-i k^{2L+1})^2]. \quad (34)$$

With the third-order λ given by (32), the higher-order τ functions $\tau_L^{(l)}$, $l > 3$ diverges in the limit $\Lambda \rightarrow \infty$. In order to obtain a finite $\tau_L^{(l)}$, $l > 3$, one should modify the coupling strength λ . A finite $\tau_L^{(l)}$ up to l th order, in the limit $\Lambda \rightarrow \infty$, can be obtained by employing the following coupling

$$\lambda_L^{(l)} = \bar{\lambda}_L \left[1 + \sum_{j=1}^{l-1} \left(\bar{\lambda}_L \frac{2}{\pi} \sum_{i=0}^L \frac{k^{2(L-i)} \Lambda^{2i+1}}{2i+1} \right)^j \right]. \quad (35)$$

With this λ , the l th-order τ function is given by

$$\tau_L^{(l)}(k) = \lambda_L^{(l)} \left[1 + \sum_{j=1}^{l-1} \lambda_L^{(l)j} I_{\text{RL}}^j(k, \Lambda) \right]. \quad (36)$$

Once the limit $\Lambda \rightarrow \infty$ is taken in (36) and terms up to the order of $\bar{\lambda}_L^l$ are maintained, the following result is obtained

$$\tau_L^{(l)}(k) = \bar{\lambda}_L \left[1 + \sum_{j=1}^l (-\bar{\lambda}_L i k^{2L+1})^j \right]. \quad (37)$$

Unlike in the case of $\lambda\phi^4$ field theory, one can calculate the result to an arbitrarily large order in perturbation theory. The summation in (37) is a geometric series and as $l \rightarrow \infty$ this series can be summed to yield

$$\tau_{\text{RL}}(k) = \frac{1}{1/\bar{\lambda}_L + i k^{2L+1}} \quad (38)$$

which is the result of perturbative renormalization.

If we compare the result of perturbative renormalization (38) with the exact renormalized solution (23) we realize that these two are equivalent if $\Lambda_{0L}(k^2)$ of (23) is identified as $-\bar{\lambda}_L^{-1}$ of (38). If this identification is made, the result of perturbative renormalization is consistent with the exact renormalized result. Then we find that the parameter $\Lambda_{0L}(k^2)$ of exact renormalization is intimately related to the strength parameter $\bar{\lambda}_L$ of perturbative renormalization.

4. Renormalization group equations

The renormalized τ function is independent of μ , so is invariant under the group of transformations $\mu \rightarrow \exp(s)\mu$, which form the RG. In the present case, as in the $\lambda\phi^4$ model, it is convenient to work in terms of the dimensionless coupling, $g_{\text{RL}}(\mu)$, defined by

$$g_{\text{RL}}(k, \mu) \equiv \mu^{2L+1} \lambda_{\text{RL}}(k, \mu). \quad (39)$$

The renormalization condition is given by

$$\mu \frac{d}{d\mu} \tau_{\text{RL}}(k, g_{\text{RL}}(k, \mu), \mu) = 0 \quad (40)$$

or,

$$\left[\mu \frac{\partial}{\partial \mu} + \beta_L(k, g_{\text{RL}}(k, \mu), \mu) \frac{\partial}{\partial g_{\text{RL}}} \right] \tau_{\text{RL}}(k, g_{\text{RL}}(k, \mu), \mu) = 0 \quad (41)$$

where

$$\beta_L(k, g_{\text{RL}}(k, \mu), \mu) = \mu \frac{\partial g_{\text{RL}}(k, \mu)}{\partial \mu}. \quad (42)$$

Equation (41) is the RG equation.

As the present problem permits analytic solution, the constant β_L of (42) can be exactly calculated. From (39) and (42) we have

$$\beta_L(k, g_{RL}(k, \mu), \mu) = (2L + 1)g_{RL}(k, \mu) + \mu^{2L+2} \frac{\partial \lambda_{RL}(k, \mu)}{\partial \mu}. \quad (43)$$

With $\lambda_{RL}(k, \mu)$ defined by (20), we have from (39) and (43)

$$\beta_L(k, g_{RL}(k, \mu), \mu) = (2L + 1)g_{RL} + g_{RL}^2 \frac{2}{\pi} \sum_{i=0}^L k^{2(L-i)} \mu^{2(i-L)}. \quad (44)$$

For $L = 0$ the β function is energy independent and depends implicitly on μ through coupling g_{RL} , whereas for $L \neq 0$ the β function has explicit dependence on both energy and μ .

The following equation expresses the invariance of the τ function $\tau_{RL}(k, g_{RL}(k, \mu), \mu)$ under a change of momentum scale:

$$\tau_{RL}(\gamma k, g_{RL}(k, \mu), \mu) = \gamma^{-(2L+1)} \tau_{RL}(k, g_{RL}(k, \mu), \mu \gamma^{-1}). \quad (45)$$

Equations (22) and (39) are consistent with scaling (45). In (45) the change of scale is effected on the explicit momentum (k) dependence of the τ function and not on the implicit momentum dependence of the coupling constant g_{RL} . From (45) we obtain

$$\left[\gamma \frac{\partial}{\partial \gamma} + \mu \frac{\partial}{\partial \mu} + (2L + 1) \right] \tau_{RL}(\gamma k, g_{RL}(k, \mu), \mu) = 0. \quad (46)$$

Eliminating the partial derivative $\mu(\partial \tau_{RL} / \partial \mu)$ between (41) and (46) we have

$$\left[\gamma \frac{\partial}{\partial \gamma} - \beta_L(k, g_{RL}(k, \mu), \mu) \frac{\partial}{\partial g_{RL}} + (2L + 1) \right] \tau_{RL}(\gamma k, g_{RL}(k, \mu), \mu) = 0 \quad (47)$$

with β_L given by (44). The RG equation (47) expresses the effect on the τ function of scaling up momentum by a factor γ .

The RG equations (41) and (47) involve the renormalized coupling g_{RL} and the renormalization scale μ and are not closely related to the physical observables. However, one can write equivalent RG equations in terms of the physical scales a_L , b_L , and c_L of (25), which are closely related to experimental observables. From (22), (25) and (39) one has the identity

$$\begin{aligned} & \left[\gamma \frac{\partial}{\partial \gamma} - \beta_L \frac{\partial}{\partial g_{RL}} \right] \tau_{RL}(\gamma k, g_{RL}, \mu) \\ &= \left[\gamma \frac{\partial}{\partial \gamma} - a_L \frac{\partial}{\partial a_L} - b_L \frac{\partial}{\partial b_L} - c_L \frac{\partial}{\partial c_L} \right] \tau_{RL}(\gamma k, a_L, b_L, c_L) \end{aligned} \quad (48)$$

so that the RG equation (47) becomes

$$\left[\gamma \frac{\partial}{\partial \gamma} - a_L \frac{\partial}{\partial a_L} - b_L \frac{\partial}{\partial b_L} - c_L \frac{\partial}{\partial c_L} + (2L + 1) \right] \tau_{RL}(\gamma k, a_L, b_L, c_L) = 0. \quad (49)$$

Equations (47) and (49) express the fact that the effect of a change in the momentum scale γ on τ_{RL} can be compensated by the effect of a change in g_{RL} or equivalently, in a_L , b_L , and c_L , respectively. In the RG equation (49) a_L , b_L , and c_L are physical scales. The RG equation (49) implies the following scaling

$$\tau_{RL}(\gamma k, a_L, b_L, c_L) = \gamma^{-(2L+1)} \tau_{RL}(k, \gamma a_L, \gamma b_L, \gamma c_L). \quad (50)$$

Hence from the knowledge of the τ function or the t matrix at a certain energy one can predict the τ function at another energy. RG equations allow one to extrapolate the τ function from one energy to another.

In principle, RG equations can be solved to yield the exact renormalized τ function. However, it is illustrative to obtain the asymptotic high-energy behaviour of this τ function from RG equations (47) or (49). At high energies $\gamma \rightarrow \infty$, and (47) or (49) reduces to

$$\gamma \frac{\partial \tau_{\text{RL}}(\gamma k)}{\partial \gamma} + (2L + 1) \tau_{\text{RL}}(\gamma k) = 0. \quad (51)$$

This has the simple solution $\lim_{\gamma \rightarrow \infty} \tau_{\text{RL}}(\gamma k) \sim 1/\gamma^{2L+1}$ which is again consistent with the τ function of (25).

The RG equations of this section yield certain general scaling properties of the renormalized τ function. Similar RG equations should be valid in general for potentials with certain renormalizable singular behaviour at short distances. The RG equations in terms of the physical scales and the associated scaling relations, for example, (49), (50) and (51), should be valid in general independent of whether the original problem had ultraviolet divergence or not. Hence such an equation should be useful in general. Obviously, such an equation could now be generalized to incorporate more physical scales.

5. Summary

We have emphasized the role of renormalization in non-relativistic quantum mechanics. Renormalization is desirable in most of quantum mechanical bound state and scattering problems, if one is interested in comparing the result of a physical model with experimental observables. Renormalization is essential in some problems exhibiting ultraviolet divergence, as in quantum field theory, in order to yield well-defined and finite observables. In both cases the final renormalized results could be expressed in terms of certain physical scales which are closely related to physical observables.

We have renormalized a potential model exhibiting ultraviolet divergence in all partial waves. As this model permits analytic solution, we have renormalized the solution exactly and also perturbatively. In the field theoretic model only perturbative renormalization is possible. As the present model permits both perturbative and exact solutions it gives us the unique opportunity to test the equivalence of the two. Such equivalence is established under very general conditions. The final renormalized result can be expressed equivalently, in terms of a renormalized coupling λ_{RL} and renormalization scale μ , or in terms of some physical scales related to observables.

Finally, we derived RG equations for the renormalized amplitudes expressed in terms of both renormalized coupling and physical scales. Though the physical content of both are identical, RG equations in terms of physical scales seem to be more useful from a practical point of view. Such RG equations in terms of physical scales are valid irrespective of the existence of the ultraviolet divergence in the original equation. These equations provide interesting scaling behaviour of the physical scattering amplitude. The RG equations are expected to be very useful in situations where the analytic solution is not known, for example, in other few- and many-body problems. The study of renormalization and RG equations in these cases will be an interesting topic for future investigation.

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References

- [1] Wilson K G 1971 *Phys. Rev. B* **4** 3184
Wilson K G and Kogut J 1974 *Phys. Rep. C* **12** 78
- [2] Ryder L H 1985 *Quantum Field Theory* (Cambridge: Cambridge University Press)
Ramond P 1981 *Field Theory: A Modern Primer* (Reading, MA: Benjamin/Cummings)
- [3] Bjorken J D and Drell S D 1965 *Relativistic Quantum Fields* (New York: McGraw-Hill)
Itzykson C and Zuber J B 1980 *Quantum Field Theory* (New York: McGraw-Hill)
- [4] Weinberg S 1995 *The Quantum Theory of Fields* (New York: Cambridge University Press)
- [5] Weinberg S 1991 *Nucl. Phys. B* **363** 3
Weinberg S 1990 *Phys. Lett.* **251B** 288
Weinberg S 1992 *Phys. Lett.* **295B** 114
- [6] Adhikari S K and Frederico T 1995 *Phys. Rev. Lett.* **74** 4572
Adhikari S K, Frederico T and Marinho R M 1996 *J. Phys. A: Math. Gen.* **29** 7157
- [7] Adhikari S K, Frederico T and Goldman I D 1995 *Phys. Rev. Lett.* **74** 487
- [8] Jackiw R 1991 *M. A. B. Beg Memorial Volume* ed A Ali and P Hoodbhoy (Singapore: World Scientific)
- [9] Manuel C and Tarrach R 1994 *Phys. Lett.* **328B** 113
Amelino-Camelia G 1994 *Phys. Lett.* **326B** 282
Amelino-Camelia G 1995 *Phys. Rev. D* **51** 2000
- [10] Gosdzinsky P and Tarrach R 1991 *Am. J. Phys.* **59** 70
Mead L R and Godines J 1991 *Am. J. Phys.* **59** 935
- [11] Beg M A B and Furlong R C 1985 *Phys. Rev. D* **31** 1370
- [12] Randeria M, Duan J M and Shieh L Y 1989 *Phys. Rev. Lett.* **62** 981
Randeria M, Duan J M and Shieh L Y 1990 *Phys. Rev. B* **41** 327
Adhikari S K and Ghosh A 1997 *Phys. Rev. B* **55** 1110
- [13] Weinberg S 1979 *Physica* **96A** 327
Witten E 1977 *Nucl. Phys. B* **122** 109
- [14] Kaplan D B, Savage M J and Wise M B 1996 *Nucl. Phys. B* **478** 629
- [15] de Araujo C F Jr, Tomio L, Adhikari S K and Frederico T 1997 *J. Phys. A: Math. Gen.* **30** 4687
- [16] Rodberg L S and Thaler R M 1967 *Introduction to the Quantum Theory of Scattering* (New York: Academic)